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LINEAR ALGEBRA

FINAL EXAM

Q1 1.8/2

1] $2x + y - 3z = \beta$ $\Rightarrow$ $\begin{bmatrix} 2 & 1 & -3 & | & \beta \\ 1 & 0 & \alpha & | & -2 \\ 1 & 1 & 0 & | & 3\beta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$x + \alpha z = -2$

$x + y = 3\beta$

(a) $\alpha = 2$ $b = 1$

$\Rightarrow \begin{bmatrix} 2 & 1 & -3 & | & 1 \\ 1 & 0 & 2 & | & -2 \\ 1 & 1 & 0 & | & 3 \end{bmatrix}$ ①=② ②=① $= \begin{bmatrix} 1 & 0 & 2 & | & -2 \\ 2 & 1 & -3 & | & 1 \\ 1 & 1 & 0 & | & 3 \end{bmatrix}$ ②=②-2① ③=③-① $= \begin{bmatrix} 1 & 0 & 2 & | & -2 \\ 0 & 1 & -7 & | & 5 \\ 0 & 1 & -2 & | & 5 \end{bmatrix}$ ③=③-②

$= \begin{bmatrix} 1 & 0 & 2 & | & -2 \\ 0 & 1 & -7 & | & 5 \\ 0 & 0 & 5 & | & 0 \end{bmatrix}$ ③=③$\times\frac{1}{5}$ $= \begin{bmatrix} 1 & 0 & 2 & | & -2 \\ 0 & 1 & -7 & | & 5 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$

$z = 0$

$y - 7z = 5$

$\Rightarrow y = 5$

$x + 2z = -2$

$x = -2$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}$ //

(b) $M = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & \alpha \\ 1 & 1 & 0 \end{bmatrix}$

$\det M = 2\begin{vmatrix} 0 & \alpha \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & \alpha \\ 1 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$\det M = 2(-\alpha) + \alpha - 3$

$= -2\alpha + \alpha - 3 \Rightarrow -\alpha - 3$

Det M cannot be 0,

so

$-\alpha - 3 = 0 \Rightarrow \alpha \neq -3$

The system has one solution for all values except for -3.

(c) $\begin{bmatrix} 2 & 1 & -3 & | & \beta \\ 1 & 0 & \alpha & | & -2 \\ 1 & 1 & 0 & | & 3\beta \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \alpha & | & -2 \\ 2 & 1 & 3 & | & \beta \\ 1 & 1 & 0 & | & 3\beta \end{bmatrix}$ ②=②-2① ③=③-① $= \begin{bmatrix} 1 & 0 & \alpha & | & -2 \\ 0 & 1 & 3-2\alpha & | & \beta+4 \\ 0 & 1 & -\alpha & | & 3\beta+2 \end{bmatrix}$ ③=③-②

$= \begin{bmatrix} 1 & 0 & \alpha & | & -2 \\ 0 & 1 & 3-2\alpha & | & \beta+4 \\ 0 & 0 & \alpha-3 & | & 2\beta-2 \end{bmatrix}$

$\alpha - 3 = 0$

$\alpha = 3$ //

$2\beta - 2 = 0$

$\beta = 1$ //

at these values, there is ~~one~~ no solution

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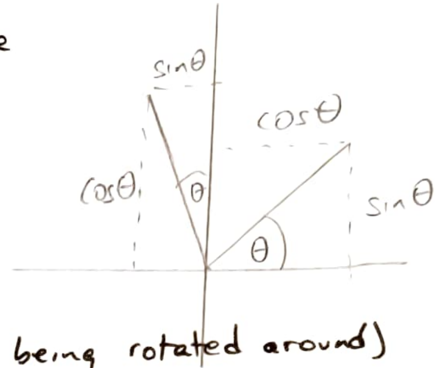
2] (a) Rotated around x_3 axis

An observer on the x_3 axis would see something like the graph on the right.



From this graph, we see that
e_1 is mapped into $(\cos\theta \;\; \sin\theta \;\; 0)^T$,
e_2 is mapped into $(-\sin\theta \;\; \cos\theta \;\; 0)^T$
and e_3 is not mapped (since it is being rotated around)

Hence, $A_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

0.4

(b) This maps it onto the $x_1 x_2$ plane, which means that x_3 must go 0.
Hence, the last row and column of the matrix A_2 must be 0.
Since it is just being mapped, there are no scalar multiples involved.
Hence, $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

0.2 in R^2

(c) $A = A_2 A_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$ //

0.4

(d) $A_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} \cos\theta-\lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

$\det(A-\lambda I) = -(1-\lambda)\begin{vmatrix} \cos\theta-\lambda & \sin\theta \\ \sin\theta & \cos\theta-\lambda \end{vmatrix}$ $= +(1-\lambda)\left[(\cos\theta-\lambda)^2 + \sin^2\theta\right]$

$= +(1-\lambda)\left[(\cos\theta^2 + \lambda^2 - 2\lambda\cos\theta + \sin^2\theta\right]$

$= +(1-\lambda)\left[\lambda^2 - 2\lambda\cos\theta + 1\right] = 0$

~~$= \cos\theta \;\; i\sin\theta \left[= (\lambda+1)\left[\lambda^2 - 2\lambda\cos\theta + 1\right] = 0\right.$~~

$\lambda - 1 = 0$
$\lambda_1 = +1$ //

$\lambda^2 - 2\lambda\cos\theta + 1 = 0$
(using quadratic formula)
$\lambda_2 = \cos\theta + i\sin\theta$ //
$\lambda_3 = \cos\theta - i\sin\theta$ //

0.3

2d] cont.d/

for $\lambda_1 = +1$

$$A - \lambda I = \left[\begin{array}{ccc|c} \cos\theta + 1 & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta + 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$(\cos\theta - 1)x_1 - (\sin\theta)x_2 = 0$$

$$(\sin\theta)x_1 + (\cos\theta - 1)x_2 = 0$$

$$x_1 = \left(\frac{\sin\theta}{\cos\theta - 1}\right)x_2$$

$$v_1 = \begin{bmatrix} \frac{\sin\theta}{\cos\theta - 1} \\ 1 \\ 0 \end{bmatrix} //$$

For $\lambda_2 = \cos\theta + i\sin\theta$

$$A - \lambda I = \left[\begin{array}{ccc|c} \cos\theta - (\cos\theta + i\sin\theta) & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta - (\cos\theta + i\sin\theta) & 0 & 0 \\ 0 & 0 & 1 - (\cos\theta + i\sin\theta) & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -i\sin\theta & -\sin\theta & 0 & 0 \\ \sin\theta & -i\sin\theta & 0 & 0 \\ 0 & 0 & 1 - (\cos\theta + i\sin\theta) & 0 \end{array}\right] \begin{array}{l} \times \frac{i}{\sin\theta} \\ \times \frac{i}{\sin\theta} \end{array} = \left[\begin{array}{ccc|c} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 - \cos\theta - i\sin\theta & 0 \end{array}\right]$$

$$x_3 = 0$$

$$x_1 - ix_2 = 0 \qquad x_1 = ix_2$$

$$ix_1 + x_2 = 0$$

$$v_2 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} //$$

v_3 is complex conjugate to

$$v_3 = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} //$$

0.3

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] $A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

a) $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & 0 \\ a_{21} & a_{22}-\lambda & 0 \\ a_{21} & a_{32} & a_{33}-\lambda \end{bmatrix}$

$\det(A - \lambda I)$

$= (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) - a_{12}(a_{21}(a_{33}-\lambda))$

$(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) - (a_{12}a_{21})(a_{33}-\lambda)$

$(a_{33}-\lambda)\left[(a_{22}-\lambda)(a_{11}-\lambda) - (a_{12}a_{21})\right] = 0$

$a_{33} - \lambda_1 = 0$

$\lambda_1 = a_{33}$

$\lambda^2 - a_{22}\lambda - a_{11}\lambda + a_{22}a_{11} - a_{12}a_{21} = 0$

$\lambda^2 - \lambda(a_{22}+a_{11}) + (a_{11}a_{22} - a_{12}a_{21}) = 0$ 0,6 pt

$\lambda_{2,3} = \dfrac{(a_{22}+a_{11}) \pm \sqrt{(a_{22}+a_{11})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}}{2}$

(b) $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 21$

$a_{11}a_{22} - a_{12}a_{21} = 21$

Let us assume

$\lambda_1 = 6 \Rightarrow a_{33} = 6$

$a_{11} + a_{22} + a_{33} = 16$

$a_{11} + a_{22} = 16 - 6 = 10$

$\lambda_{2,3} = \dfrac{10 \pm \sqrt{10^2 - 4(21)}}{2}$ → substituting values calculated into answer from (a)

$\lambda_2 = \dfrac{10 + \sqrt{16}}{2}$ $\lambda_3 = \dfrac{10 - \sqrt{16}}{2}$

$\lambda_2 = 7$ // $\lambda_3 = 3$ // 0,6 pt

(c) I will now re-order the eigenvalues in order for the corresponding eigenvectors to be easily understood for me.

$\lambda_1 < \lambda_2 < \lambda_3 \Rightarrow \lambda_1 = 7$, $\lambda_2 = 6$, $\lambda_3 = 3$

$P = [v_1 \,\vdots\, v_2 \,\vdots\, v_3]$

$\Rightarrow P = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 0 & 1 \\ 7 & 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A = PDP^{-1}$

so, I will first find P^{-1}

3c]

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 7 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} ① = ①/2 \\ ② = ② + ① \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 7 & 1 & 1 & 0 & 0 & 1 \end{array}\right] ③ = ③ - 7① = \left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & -4/3 & -7/3 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & -4/3 & -7/3 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array}\right] ③/2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & -4/3 & -7/3 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array}\right] ① = ① - \tfrac{1}{3}③$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 1 & -4/3 & -7/3 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array}\right] ② = ② + \tfrac{4}{3}③ = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & -1/6 & 0 \\ 0 & 1 & 0 & -5/3 & 2/3 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & 0 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & 0 \\ -\frac{5}{3} & \frac{2}{3} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$ ✓

a7pt

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 0 & 1 \\ 7 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & 0 \\ -5/3 & 2/3 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 21 & 0 & 3 \\ -21 & 0 & 3 \\ 49 & 6 & 3 \end{bmatrix} \begin{bmatrix} 1/6 & -1/6 & 0 \\ -5/3 & 2/3 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 5 & 0 \\ -1/3 & -8/3 & 6 \end{bmatrix} //$$

1.6 pts

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4] $x_1'(t) = 2x_1(t) + 3x_2(t)$ $\quad x_1(0) = 4$

$x_2'(t) = -2x_1(t) + 6x_2(t)$ $\quad x_2(0) = 1$

(a) $A = \begin{bmatrix} 2 & 3 \\ -2 & 6 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ -2 & 6-\lambda \end{bmatrix}$

$\det(A-\lambda I) = 0$

$(2-\lambda)(6-\lambda) + 6 = 0$

$\lambda^2 + 12 - 8\lambda + 6 = 0 \Rightarrow \lambda^2 - 8\lambda + 18 = 0$

$\lambda_{1,2} = \dfrac{8 \pm \sqrt{(-8)^2 - 4(18)}}{2} \Rightarrow \lambda_1 = \dfrac{8 + \sqrt{-8}}{2} \Rightarrow \lambda_1 = \dfrac{8 + (2\sqrt{2})i}{2}$ #

$\lambda_1 = 4 + \sqrt{2}i$ // and $\lambda_2 = 4 - \sqrt{2}i$ //

for λ_1

$A - \lambda I = \begin{bmatrix} 2-(4+\sqrt{2}i) & 3 \\ -2 & 6-(4+\sqrt{2}i) \end{bmatrix} = \left[\begin{array}{cc|c} -2+\sqrt{2}i & 3 & 0 \\ -2 & 2+\sqrt{2}i & 0 \end{array}\right]$ $\begin{matrix} ① = ① \times (-2-\sqrt{2}i) \\ \text{~~② = ② × (2-√2i)~~} \end{matrix}$

$= \left[\begin{array}{cc|c} 6 & -6+3\sqrt{2}i & 0 \\ -2 & 2+\sqrt{2}i & 0 \end{array}\right]$ $\begin{matrix} ① = ①/3 \\ ② = ② + \frac{1}{3}① \end{matrix}$ $= \left[\begin{array}{cc|c} 2 & -2+\sqrt{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$2x_1 + (-2+\sqrt{2}i)x_2 = 0$

$2x_1 = (2-\sqrt{2}i)x_2$

$x_1 = \left(\dfrac{2-\sqrt{2}i}{2}\right)x_2$

$v_1 = \begin{bmatrix} 2-\sqrt{2}i \\ 2 \end{bmatrix}$ // (choosing $x_2 = 2$)

λ_2 is a complex conjugate, so $v_2 = \begin{bmatrix} 2+\sqrt{2}i \\ 2 \end{bmatrix}$ //

4b] ~~continued~~

~~$x_1 = e^{4t} v_1$~~

~~$x_1 = e^{4+\sqrt{2}it}$~~ $v_2 =$

Using formula from book

$$x_1 = e^{4t}\left\{(\cos\sqrt{2}t)\begin{bmatrix}2\\2\end{bmatrix} - (\sin\sqrt{2}t)\begin{bmatrix}\sqrt{2}i\\0\end{bmatrix}\right\} \qquad x_2 = e^{4t}\left\{(\cos\sqrt{2}t)\begin{bmatrix}\sqrt{2}i\\0\end{bmatrix} + (\sin\sqrt{2}t)\begin{bmatrix}2\\2\end{bmatrix}\right\}$$

$$x_1 = e^{4t}\left\{\begin{bmatrix}2\cos(\sqrt{2}t)\\2\cos(\sqrt{2}t)\end{bmatrix} - \begin{bmatrix}i\sqrt{2}\sin(\sqrt{2}t)\\0\end{bmatrix}\right\} \qquad x_2 = e^{4t}\left\{\begin{bmatrix}i\sqrt{2}\cos\sqrt{2}t\\0\end{bmatrix} + \begin{bmatrix}2\sin\sqrt{2}t\\2\sin\sqrt{2}t\end{bmatrix}\right\}$$

$$x_1 = e^{4t}\begin{bmatrix}2\cos\sqrt{2}t - i\sqrt{2}\sin\sqrt{2}t\\2\cos\sqrt{2}t\end{bmatrix} \qquad x_2 = e^{4t}\begin{bmatrix}i\sqrt{2}\cos\sqrt{2}t + 2\sin\sqrt{2}t\\2\sin\sqrt{2}t\end{bmatrix}$$

Real solution only, so

$$Re(x) = e^{4t}\begin{bmatrix}2\cos\sqrt{2}t\\2\cos\sqrt{2}t\end{bmatrix} \qquad Re(x_2) = e^{4t}\begin{bmatrix}2\sin\sqrt{2}t\\2\sin\sqrt{2}t\end{bmatrix}$$

$$Re(x) = c_1 Re(x_1) + c_2 Re(x_2)$$

$$\left[\begin{array}{cc|c}2 & 0 & 4\\2 & 0 & 1\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}1 & 0 & 2\\1 & 0 & 1/2\end{array}\right]$$

$$c_2 = 0$$
$$c_1 = 2$$

$$Re(x) = c_1 Re(x_1)$$

$$Re(x) = 2e^{4t}\begin{bmatrix}2\cos\sqrt{2}t\\2\cos\sqrt{2}t\end{bmatrix} \quad //$$

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5] (a) $v_1 = \begin{bmatrix} 2 \\ a \\ b \end{bmatrix}$ $\hat{v}_1 = \frac{v_1}{|v_1|}$ $|v_1| = \sqrt{2^2+a^2+b^2} = \sqrt{4+a^2+b^2}$

$\hat{v}_1 = \frac{1}{\sqrt{4+a^2+b^2}} \begin{bmatrix} 2 \\ a \\ b \end{bmatrix}$ //

0.3

(b) $a=1$ $b=2$

$\Rightarrow v_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\hat{v}_1 = \frac{1}{\sqrt{4+a^2+b^2}} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \hat{v}_1 = \frac{1}{\sqrt{4+1+4}} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \hat{v}_1$ //

0.3

(c) Let $v_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ when it is orthogonal,

$v_1 \cdot v_2 = 0$

$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow 2x_1 + x_2 + 2x_3 = 0$ // ~~$\Rightarrow \begin{bmatrix} 2d \\ e \\ 2f \end{bmatrix}$ where d, e, f are scalars following the general rule~~

solve eq for one of the variables in terms of the others

0.2

(d) For this, I will choose a vector that satisfies the general form above, so,

let ~~v_2 =~~ ~~2~~ $v_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$\hat{v}_2 = \frac{v_2}{|v_2|}$ $|v_2| = \sqrt{1^2+(-2)^2+0^2} = \sqrt{1+4} = \sqrt{5}$

$\hat{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ //

0.3

(e) Any scalar multiplied with the unit vector would give a vector in the direction.

Hence,

$v_3 = \sqrt{5} \times \hat{v}_2$

$v_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ is in the direction of v_2.

0.3

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f) The cross product of v_1 and v_2 would result in another vector which is orthogonal to both v_1 and v_2.
I will call this new vector v_4

$$V_4 = V_1 \times V_2$$

$$V_1 \times V_2 = \begin{vmatrix} x & y & z \\ 2 & a & b \\ 2d & e & 2f \end{vmatrix} = x(2af - be) - y(4f - 2bd) + z(2e - 2ad)$$

Into ~~vector form~~ column vector form

$$\Rightarrow \begin{bmatrix} 2af - be \\ 2bd - 4f \\ 2e - 2ad \end{bmatrix}$$

0.4

where a, b, c, d, e, f are variables as defined in previous solutions.